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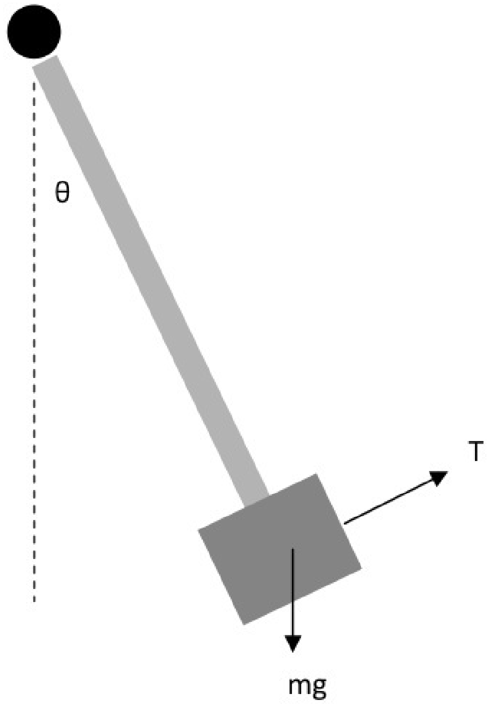
Final Project

A ME 455
November 17, 2008

The design objective is to design and implement a controller that will command the pendulum to move to 50^o with a 5% settling time not exceeding 2 seconds and a steady-state error not exceeding 10%, (5^o).



1. **Submit second order differential equation for** θ.

   Solution. Newton's 2^{nd} law of rotation is

   $$\tau \;=\; I\alpha.$$

   Apply this relationship to the nonlinear pendulum. Let T be the applied force, $c\dot{\theta}$ is the frictional torque at the pivot point. Since the force acts perpendicularly to the rod, $\tau_{\text{input}} = Tl$. Combining all torques at the pivot point gives the net torque at the pivot

   $$ml^2\ddot{\theta} \;=\; -mgl\sin\theta - c\dot{\theta} + Tl.$$

   Reduce the equation by getting rid of common terms to get the final result

   $$\ddot{\theta} + \frac{c}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta = \frac{1}{ml}T. \tag{1}$$

2. **Submit transfer function between u and θ.**

   Solution. Let $T = mg\sin\theta + u$. Substitute into (1) to get

$$\begin{aligned}\ddot{\theta} + \frac{c}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta &= \frac{1}{ml}\Big(mg\sin\theta + u\Big)\\ \implies \ddot{\theta} + \frac{c}{ml^2}\dot{\theta} &= \frac{1}{ml}u.\end{aligned}$$

   Take the Laplace transform to get the expression

$$(s^2\Theta(s) - s\theta(0) - \dot{\theta}(0)) + \frac{c}{ml^2}(s\Theta(s) - \theta(0)) = \frac{1}{ml}U(s).$$

   Assume the initial conditions are $\theta(0) = \dot{\theta}(0) = 0$, since the system starts at rest from the bottom of its arc. The transfer function is

$$\frac{\Theta(s)}{U(s)} = \frac{\frac{1}{ml}}{s\big(s + \frac{c}{ml^2}\big)}. \tag{2}$$

3. **Submit (1) expression for the parameter S in terms of the physical parameters of the system including the motor proportionality constant K and (2) the transfer function between $u(t)$ and θ containing the parameters of the system.**

   Solution. For (1), start with $T = K(\tilde{u} - u_0)$, where u_0 is the lower threshold voltage required to overcome motor friction. Now, to solve for S, consider

$$\begin{aligned}T &= K(\tilde{u} - u_0)\\ &= K(S\sin\theta + Ku(t)).\end{aligned}$$

   Plug into eq. (1) to get

$$\ddot{\theta} + \frac{c}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta = K\frac{g}{l}\sin\theta + \frac{K}{ml}u(t).$$

   In order to cancel the $\sin\theta$ term,

$$S = \frac{mg}{K}. \tag{3}$$

   For (2), the steps follow almost exactly as before in part 2.

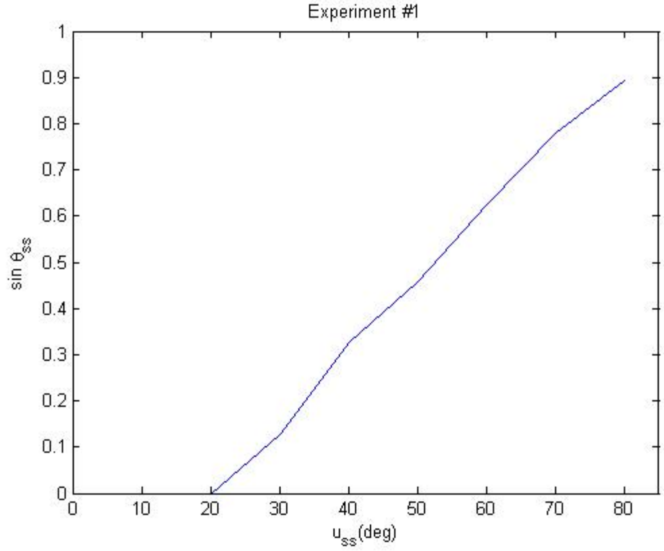
$$(s^2\Theta(s) - s\theta(0) - \dot{\theta}(0)) + \frac{c}{ml^2}(s\Theta(s) - \theta(0)) = \frac{K}{ml}U(s).$$

   Assume the initial conditions are $\theta(0) = \dot{\theta}(0) = 0$, since the system starts at rest from the bottom of its arc. The transfer function is

$$\frac{\Theta(s)}{U(s)} = \frac{\frac{K}{ml}}{s\big(s + \frac{c}{ml^2}\big)}. \tag{4}$$

4. **Experiment #1: Submit experimental plot for** $\sin\theta_{ss}$ **and** $\tilde{u}_{ss}$ **and value for parameters** $u_0, K/mg$**, and** S**.**

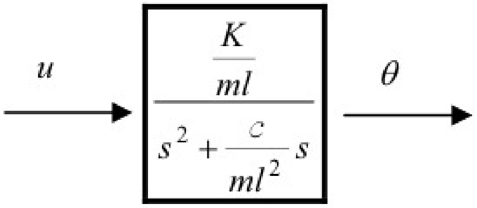
   Solution. The following plot shows open-loop response for several values of $\tilde{u}_{ss}$.



   From the plot, I calculate the following parameters

$$\begin{aligned} u_0 &= 20^o \\ \frac{K}{mg} &= \arctan\left(\frac{0.8063-0}{70-20}\right) = 0.0161 \\ S &= \frac{mg}{Kl} = \frac{1}{0.0161} = 62.01. \end{aligned}$$

5. It is clear that the system can now be described by the open loop transfer function



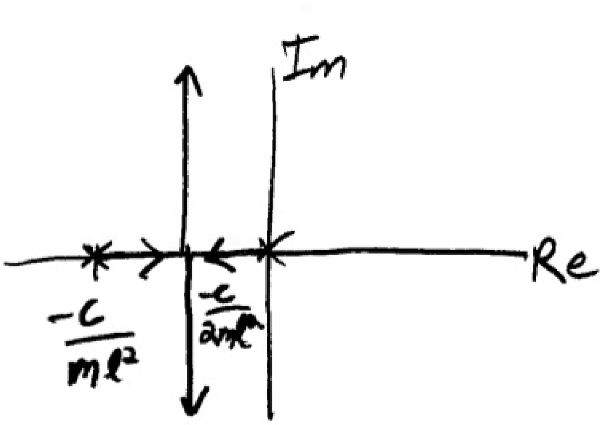
   Q1: Do you expect that the closed loop system will be stable for all values of the proportional gain K_p? **Submit sketched root locus and answer to Q1.**

   Solution. First, calculate the poles for the transfer function

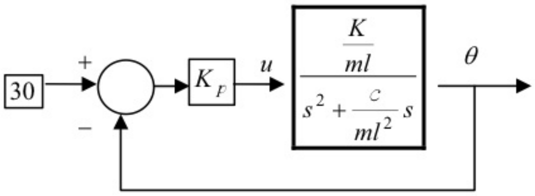
$$GH = \frac{\frac{K_pK}{ml}}{s^2 + \frac{c}{ml^2}s}.$$

The poles are $s = 0, -\frac{c}{ml^2}$. The asymptote location and the breakout point are co-located at $s = -\frac{c}{2ml^2}$.

The root locus sketch for the system is

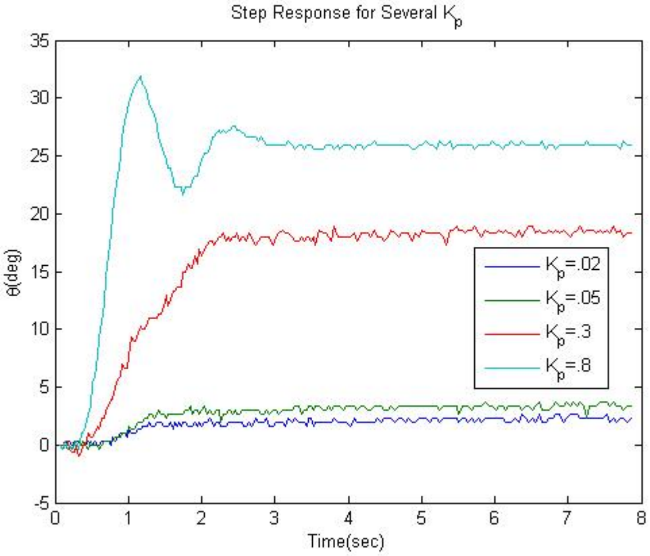


Given the root locus sketch, I expect that the system is stable for all K_p.



Experiment #2: Submit plots for step response using gain $K_p = 0.02; 0.05; 0.3; 0.8$. From the plots estimate the steady-state error for each run and report it. What is the type of the above system based on its transfer function? What is its expected steady-state error behavior from your experiments? Do you observe the expected stability behavior from your experiments? For at least one of your simulations, induce disturbances by slightly pushing on the rod and letting it go. Does the system recover? For what gain is the recovery faster? On each plot, mark the gain used.

Solution. The following plot shows the system step response given values of K_p,



Given the effective angle command, u, the following steady-state values are calculated

K_p	u (deg)	$\theta_{ss}(deg)$	$e_{ss}(deg)$
0.02	3.0713	2.6599	0.4114
0.05	5.2721	3.6574	1.6147
0.30	22.9680	18.9000	4.0680
0.80	30.3676	25.9341	4.4335

From the transfer function, the system is type 1, which has zero steady-state error, e_{ss}. However, from the plot, it is clear that all of the step responses have non-zero e_{ss}. From the root locus sketch from step 5, it becomes clear that as the gain increases the system becomes type 0, which has steady-state error $1/K_p$. Clearly, this is an inverse relationship, but the table above shows a direct relationship; which implies that the control law must contribute a term which affects the steady-state error in a direct fashion.

The system appears unconditionally stable; none of the trajectories diverge. Through the introduction of small disturbances, it is clear that all K_p result in system recovery, although the value for is slightly different as calculated in *Closed_loop.m*. It is difficult to repeat the experiment for all K_p, since each disturbance introduced is slightly different than the previous disturbance. However, it seems as though $K_p = 0.8$ recovers fastest. For significantly large initial disturbances, something interesting happens: the system begins to show signs of instability; *i.e.* the system shows slow exponential growth.

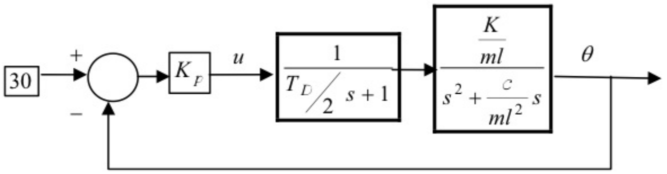


Figure 1: Closed-loop block diagram

6. **Submit root locus, critical value of the gain K_p^* and explanation of how step 6 explains experimental observations from step 5.**

   Solution. Taking the mean of the time steps $\Delta t(i-1) = t(i) - t(i-1)$, $i = 2, 200$ yields the discrete sampling time $T_D = 0.0404$. Plugging this value into the transfer functions from the above block diagram (Figure 1), along with $K/ml = 33$ and $c/ml^2 = 0.7$, the transfer function becomes

$$GH = \frac{33K_p}{s(0.0202s+1)(s+0.7)}$$

   The resulting root locus is

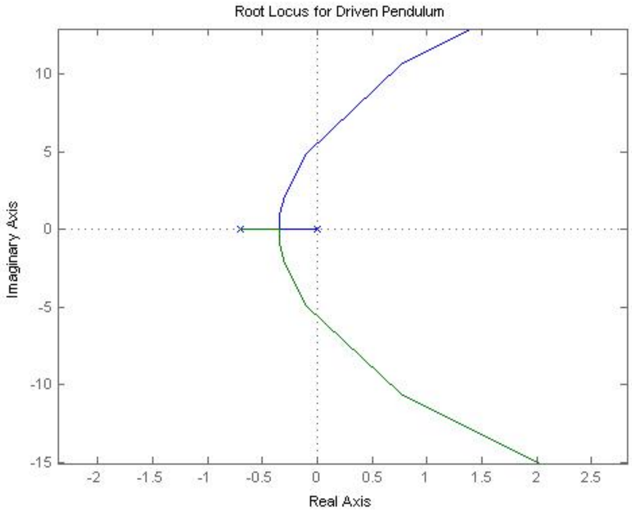


Figure 2: Root Locus

which does indeed go unstable. I have chosen the axes such that the third pole is not visible because this pole's contribution to the natural frequency and damping coefficient is negligible. The fact that the system does show signs of instability could be the explanation for why introducing large disturbances makes the system goes unstable.

This would indeed explain why I observe the stability margin decrease as the commanded angle increases. Glancing at the plot though, the system does not go unstable until a very large gain, $K_p^* \approx 46.4$. However, this gain will have absorbed in it the factors from the control law, such as the large $S = 62.01$. So, the actual gain to be used when commanding will be much smaller than this critical value.

7. **Submit the value of K_p satisfying the design requirements and the experimental step response. Mark on the plot the experimental steady-state error and the 5% settling time, demonstrating that the design requirements have been met.**

   Solution. First off, I need to define the quantitative requirements to be met for $\theta_0 = 50^o$:

$$\begin{aligned} T_s &\leq \frac{3}{\zeta\omega_n} \text{ which must also be less than 2 sec.} \\ e_{ss} &\leq 5^o. \end{aligned}$$

   Rewrite the equation for the two dominant poles

$$a_1, a_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

   Now, use the design constraint $T_s \leq 2$ sec to define the requirement $\zeta\omega_n \leq 1.5$. Through iterations over several gains, $K_p = 0.59$ was shown to meet the design constraints. The figure below shows the system step response.